Chapter 1. Relations and Functions

Concept of Relation and Functions

1 Mark Questions

1. If $R = \{(a, a^3) : a$ is a prime number less than $5\}$ be a relation. Find the range of R.

Foreign 2014

Given, $R = \{(a, a^3)$: a is a prime number less than $5\}$

We know that, 2 and 3 are the prime numbers less than 5.

$\therefore$ a can take values 2 and 3.

Then, $R = \{(2, 2^3), (3, 3^3)\} = \{(2, 8), (3, 27)\}$

Hence, the range of R is $\{8, 27\}$. **(1)**

2. If $f: \{1,3,4\} \to \{1,2,5\}$ and $g: \{1,2,5\} \to \{1,3\}$ given by $f = \{(1,2), (3,5), (4,1)\}$ and $g = \{(1,3), (2,3), (5,1)\}$. Write down gof.

All India 2014C

The functions $f: \{1,3,4\} \to \{1,2,5\}$ and $g: \{1,2,5\} \to \{1,3\}$ are defined as

$f = \{(1,2), (3,5), (4,1)\}$ and $g = \{(1,3), (2,3), (5,1)\}$

$$\therefore \quad gof(1) = g\{f(1)\} = g(2) = 3 \quad [\because f(1) = 2 \text{ and } g(2) = 3]$$

$$gof(3) = g(f(3)) = g(5) = 1 \quad [\because f(3) = 5 \text{ and } g(5) = 1]$$

$$gof(4) = g(f(4)) = g(1) = 3 \quad [\because f(4) = 1 \text{ and } g(1) = 3]$$

$$\therefore \quad gof = \{(1,3), (3,1), (4,3)\}$$ **(1)**

3. Let R is the equivalence relation in the set $A = \{0,1,2,3,4,5\}$ given by $R = \{(a, b): 2$ divides $(a - b)\}$. Write the equivalence class $[0]$.

Delhi 2014C

. Given, $R = \{(a, b) : 2 \text{ divides } (a-b)\}$

Here, all even integers are related to zero, i.e. $(0, 2)(0, 4)$.

Hence, equivalence class of $[0] = \{2, 4\}$ **(1)**

4. If $R = \{(x, y) : x + 2y = 8\}$ is a relation on N, then write the range of R. **All India 2014**

Given, the relation R is defined on the set of natural numbers, i.e. N as

$$R = \{(x, y) : x + 2y = 8\}$$

To find the range of R, $x + 2y = 8$ can be rewritten as $y = \dfrac{8 - x}{2}$.

On putting $x = 2$, we get $y = \dfrac{8-2}{2} = 3$

On putting $x = 4$, we get $y = \dfrac{8-4}{2} = 2$

On putting $x = 6$, we get $y = \dfrac{8-6}{2} = 1$

As, $x, y \in N$, then $R = \{(2, 3)\ (4, 2)\ (6, 1)\}$

Hence, range of relation is $\{3, 2, 1\}$. **(1)**

5. If $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$ and $f = \{(1, 4), (2, 5), (3, 6)\}$ is a function from A to B. State whether f is one-one or not.

All India 2011

Given, $A = \{1, 2, 3\}$ and $B = \{4, 5, 6, 7\}$

Now, $f : A \rightarrow B$ is defined as

$$f = \{(1, 4), (2, 5), (3, 6)\}$$

Therefore, $f(1) = 4$, $f(2) = 5$ and $f(3) = 6$.

It is seen that the images of distinct elements of A under f are distinct. So, f is one-one. **(1)**

6. If $f : R \rightarrow R$ is defined by $f(x) = 3x + 2$, then define $f[f(x)]$. **Foreign 2011; Delhi 2010**

Given, $f(x) = 3x + 2$

Now, $f[f(x)] = f(3x + 2) = 3(3x + 2) + 2$

$$= 9x + 6 + 2 = 9x + 8$$

7. Write fog, if $f : R \rightarrow R$ and $g : R \rightarrow R$ are given by $f(x) = |x|$ and $g(x) = |5x - 2|$. **Foreign 2011**

Given, $f(x) = |x|, g(x) = |5x - 2|$

Now, $fog\ (x) = f[g(x)] = f\{|5x - 2|\}$

$$= ||5x - 2|| = |5x - 2|$$

8. Write fog, if $f : R \rightarrow R$ and $g : R \rightarrow R$ are given by $f(x) = 8x^3$ and $g(x) = x^{1/3}$. **Foreign 2011**

Given, $f(x) = 8x^3$ and $g(x) = x^{1/3}$

Now, $fog\ (x) = f[g(x)] = f(x^{1/3}) = 8(x^{1/3})^3 = 8x$ **(1)**

9. State the reason for the relation R in the set $\{1, 2, 3\}$ given by $R = \{(1, 2), (2, 1)\}$ not to be transitive. **Delhi 2011**

We know that, for a relation to be transitive
$(x, y) \in R$ and $(y, z) \in R$

$\Rightarrow \quad (x, z) \in R$

Here, $(1, 2) \in R$ and $(2, 1) \in R$ but $(1, 1) \notin R$.
Hence, R is not transitive. **(1)**

10. What is the range of the function
$f(x) = \frac{|x-1|}{x-1}, x \neq 1$? Delhi 2010; HOTS

Given, function is $f(x) = \frac{|x-1|}{x-1}, x \neq 1$

The above function may be written as

$$f(x) = \begin{cases} \frac{x-1}{x-1}, & \text{if } x > 1 \\ -\frac{(x-1)}{x-1}, & \text{if } x < 1 \end{cases}$$

$$\Rightarrow \quad f(x) = \begin{cases} 1, & \text{if } x > 1 \\ -1, & \text{if } x < 1 \end{cases}$$

$\therefore$ Range of $f(x)$ is the set $\{-1, 1\}$.

11. If $f: R \to R$ is defined by $f(x) = (3 - x^3)^{1/3}$, then find $fof(x)$. All India 2010

Given, function is $f: R \to R$ such that $f(x) = (3 - x^3)^{1/3}$.

Now, $fof\,(x) = f[f(x)] = f[(3 - x^3)^{1/3}]$

$$= [3 - \{(3 - x^3)^{1/3}\}^3]^{1/3}$$

$$= [3 - (3 - x^3)]^{1/3} = (x^3)^{1/3}$$

$$= x \qquad \textbf{(1)}$$

12. If f is an invertible function, defined as

$f(x) = \dfrac{3x - 4}{5}$, then write $f^{-1}(x)$. **Foreign 2010**

Given, $f(x) = \dfrac{3x - 4}{5}$ and is invertible.

Let $\quad y = \dfrac{3x - 4}{5} \Rightarrow 5y = 3x - 4$

$\Rightarrow \quad 3x = 5y + 4 \Rightarrow x = \dfrac{5y + 4}{3}$

$\therefore \quad f^{-1}(y) = \dfrac{5y + 4}{3} \Rightarrow f^{-1}(x) = \dfrac{5x + 4}{3}$

13. If $f: R \to R$ and $g: R \to R$ are given by $f(x) = \sin x$ and $g(x) = 5x^2$, then find $gof(x)$.

Foreign 2010

Given, $f(x) = \sin x$ and $g(x) = 5x^2$.

$\therefore \quad gof(x) = g[f(x)] = g(\sin x)$

$$= 5(\sin x)^2 = 5 \sin^2 x$$

14. If $f(x) = 27x^3$ and $g(x) = x^{1/3}$, then find $gof(x)$.

Foreign 2010

Given, $f(x) = 27x^3$ and $g(x) = x^{1/3}$.

$$\therefore \quad gof(x) = g[f(x)] = g(27x^3)$$

$$= (27x^3)^{1/3} = (27)^{1/3} \cdot (x^3)^{1/3}$$

$$= (3^3)^{1/3} \cdot (x^3)^{1/3} = 3x$$

$$\therefore \quad gof(x) = 3x$$

15. If the function $f : R \to R$, defined by $f(x) = 3x - 4$ is invertible, then find f^{-1}.

All India 2010C

Given, function is $f(x) = 3x - 4$ and is invertible.

Let $\quad y = 3x - 4 \Rightarrow 3x = y + 4$

$$\Rightarrow \quad x = \frac{y+4}{3}$$

$$\therefore \quad f^{-1}(y) = \frac{y+4}{3} \Rightarrow f^{-1}(x) = \frac{x+4}{3} \quad \textbf{(1)}$$

16. If $f : R \to R$ defined by $f(x) = \dfrac{3x+5}{2}$ is an invertible function, then find $f^{-1}(x)$

.

All India 2009C

Do same as Que 12. $\left[\textbf{Ans. } \dfrac{2x-5}{3}\right]$

17. State whether the function $f: N \to N$ given by $f(x) = 5x$ is injective, surjective or both.

All India 2008C; HOTS

For injective function, it should be one-one and for surjective function, it should be onto.

Given function is $f(x) = 5x$.

As, $\quad f(x_1) = f(x_2) \Rightarrow 5x_1 = 5x_2$

$\Rightarrow \quad x_1 = x_2, \ \forall \ x_1, x_2 \in N$

So, $f(x)$ is an injective function. **(1/2)**

Also, range of $f(n) = 5n$, $n \in N$.

But codomain $= N$

$\because \quad$ Range $\neq$ Codomain

$\therefore f(x)$ is not surjective.

Hence, the given function is injective.

18. If $f: R \to R$ defined by $f(x) = \dfrac{2x-7}{4}$ is an invertible function, then find $f^{-1}(x)$.

Delhi 2008C

Do same as Que. 12. $\left[\textbf{Ans. } \dfrac{4x+7}{2}\right]$

4 Marks Questions

19. If $f: W \to W$, is defined as $f(x) = x - 1$, if x is odd and $f(x) = x + 1$, if x is even. Show that f is invertible. Find the inverse of f, where W is the set of all whole numbers. **Foreign 2014**

$f: W \to W$ is defined as

$$f(x) = \begin{cases} x-1, \text{ if } x \text{ is odd} \\ x+1, \text{ if } x \text{ is even} \end{cases}$$

First, we need to show that f is one-one.

Let $f(x_1) = f(x_2)$

Case I When x_1 and x_2 are odd.

Then, $\quad f(x_1) = f(x_2) \Rightarrow x_1 - 1 = x_2 - 1$

$\Rightarrow \quad x_1 = x_2, \forall x_1, x_2 \in W$ (1)

Case II When x_1 and x_2 are even.

Then, $\quad f(x_1) = f(x_2)$

$\Rightarrow \quad x_1 + 1 = x_2 + 1$

$\Rightarrow \quad x_1 = x_2, \forall x_1, x_2 \in W$

So, from case I and II, we observe that

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \in W$$

Hence, $f(x)$ is a one-one function. **(1)**

Now, we need to show that f is onto.

Any odd number $2y+1$, in the codomain W, is the image of $2y$ in the domain W.

Also, any even number $2y$ in the codomain W, is the image of $2y-1$ in the domain W.

Thus, every element in W (codomain) has its image in W (domain).

So, f is onto. Therefore, f is bijection. So, it is invertible. **(1)**

Let $x, y \in W$, such that

$$f(x) = y$$

$\Rightarrow \quad x - 1 = y$, if x is odd

$\quad x + 1 = y$, if x is even

$$\Rightarrow \quad x = \begin{cases} y+1, & \text{if } y \text{ is even} \\ y-1, & \text{if } y \text{ is odd} \end{cases}$$

Clearly, $f = f^{-1}$ **(1)**

20. If $f, g : R \to R$ are two functions defined as $f(x) = |x| + x$ and $g(x) = |x| - x, \forall\, x \in R$, Then, find fog and gof. **All India 2014C**

Given, $f(x) = |x| + x$ and $g(x) = |x| - x$ for all $x \in R$.

$$\Rightarrow f(x) = \begin{cases} 2x, & x > 0 \\ 0, & x < 0 \end{cases} \text{ and } g(x) = \begin{cases} 0, & x > 0 \\ -2x, & x < 0 \end{cases} \quad \textbf{(1)}$$

Thus, for $x > 0$, $gof(x) = g(2x) = 0$

and for $x < 0$, $gof(x) = g(0) = 0$

$\Rightarrow$ $gof(x) = 0, \forall\, x \in R$ **(1½)**

and for $x > 0$, $fog(x) = f(0) = 0$

for $x < 0$, $fog(x) = f(-2x) = -4x$

$$\Rightarrow fog(x) = \begin{cases} 0, & x > 0 \\ -4x, & x < 0 \end{cases} \quad \textbf{(1½)}$$

21. If R is a relation defined on the set of natural numbers N as follows:

$R = \{(x, y), x \in N, Y \in N \text{ and } 2x + y = 24\}$, then find the domain and range of the relation R. Also, find if R is an equivalence relation or not. **Delhi 2014C**

Given $R = \{(x, y), x \in N, y \in N \text{ and } 2x + y = 24\}$

When, $x = 1 \Rightarrow y = 22$; $x = 2 \Rightarrow y = 20$

$x = 3 \Rightarrow y = 18$; $x = 4 \Rightarrow y = 16$

$x = 5 \Rightarrow y = 14$; $x = 6 \Rightarrow y = 12$

$x = 7 \Rightarrow y = 10$; $x = 8 \Rightarrow y = 8$

$x = 9 \Rightarrow y = 6$; $x = 10 \Rightarrow y = 4$

$x = 11 \Rightarrow y = 2$

So, domain of $R = \{1, 2, 3, \ldots, 11\}$.

and range of $R = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22\}$

and $R = \{(1,22)(2,20)(3,18)(4,16)(5,14)(6,12)(7,10)(8,8)(9,6)(10,4)(11,2)\}$ **(1)**

Reflexive

Since, for $a \in$ domain of R, $(a, a) \notin R$.

Hence, R is not reflexive. **(1)**

Symmetric

Since, $(1,22) \in R$ but $(22,1) \notin R$.

Hence, R is not symmetric **(1)**

Transitive

There are no elements such that that $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$. Hence, R is not transitive and so, it is not an equivalence relation. **(1)**

22. If $A = R - \{3\}$ and $B = R - \{1\}$. Consider the function $f : A \to B$ defined by $f(x) = \dfrac{x-2}{x-3}$ for all $x \in A$. Then show that f is bijective. Find $f^{-1}(x)$. **Delhi 2014C; Delhi 2012**

Given, function is $f : A \to B$, where $A = R - \{3\}$ and $B = R - \{1\}$, such that $f(x) = \dfrac{x-2}{x-3}$.

One-one Let $f(x_1) = f(x_2)$, $\forall\ x_1, x_2 \in A$

$$\Rightarrow \quad \frac{x_1 - 2}{x_1 - 3} = \frac{x_2 - 2}{x_2 - 3}$$

$$\Rightarrow \quad (x_1 - 2)(x_2 - 3) = (x_2 - 2)(x_1 - 3)$$

$$\Rightarrow \quad x_1x_2 - 3x_1 - 2x_2 + 6 = x_1x_2 - 3x_2 - 2x_1 + 6$$

$$\Rightarrow \quad -3x_1 - 2x_2 = -3x_2 - 2x_1$$

$$\Rightarrow \quad -3(x_1 - x_2) + 2(x_1 - x_2) = 0$$

$$\Rightarrow \quad -(x_1 - x_2) = 0$$

$$\Rightarrow \quad x_1 - x_2 = 0 \Rightarrow x_1 = x_2$$

$\because$ $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$, $\forall\ x_1, x_2 \in A$. So, $f(x)$ is a one-one function. **(1½)**

Onto To show $f(x)$ is onto, we show that range of $f(x)$ and its codomain are same.

Now, let $y = \frac{x-2}{x-3} \Rightarrow xy - 3y = x - 2$

$\Rightarrow \quad xy - x = 3y - 2 \Rightarrow x(y-1) = 3y - 2$

$\Rightarrow \quad x = \frac{3y-2}{y-1}$ ...(i)

Since, $x \in R - \{3\}, \forall \ y \in R - \{1\}$, so range of $f(x) = R - \{1\}$.

Also, given codomain of $f(x) = R - \{1\}$

$\therefore$ Range = Codomain

Hence, $f(x)$ is an onto function. **(1½)**

Therefore, $f(x)$ is an bijective function.

From Eq. (i), we get

$$f^{-1}(y) = \frac{3y-2}{y-1} \Rightarrow f^{-1}(x) = \frac{3x-2}{x-1}$$

which is the inverse function of $f(x)$. **(1)**

23. If $A = \{1, 2, 3, \dots, 9\}$ and R be the relation in $A \times A$ defined by $(a, b) \, R \, (c, d)$. If $a + d = b + c$ for $(a, b), (c, d)$ in $A \times A$. Prove that R is an equivalence relation, Also, obtain the equivalence class $[(2, 5)]$. **Delhi 2014**

Given, relation R defined by $(a, b) \, R \, (c, d)$, if $a + d = b + c$ for $(a, b), (c, d)$ in $A \times A$.

Here, $A = \{1, 2, 3, \dots, 9\}$

We observe the following properties on R:

Reflexive Let $(1, 2)$ be an element of $A \times A$.

Then, $(1, 2) \in A \times A \Rightarrow 1, 2 \in A$

$\Rightarrow \quad 1 + 2 = 2 + 1$ [$\because$ addition is commutative]

$\Rightarrow \quad (1, 2) \, R \, (1, 2)$

Thus, $(1, 2) \, R \, (1, 2), \forall \ (1, 2) \in A \times A$

So, R is reflexive on $A \times A$. **(1)**

Symmetric Let $(1, 2), (3, 4) \in A \times A$ such that $(1, 2) R (3, 4)$

Then, $\quad 1 + 4 = 2 + 3$

$\Rightarrow \quad 3 + 2 = 4 + 1 \quad [\because$ addition is commutative]

$\Rightarrow \quad (3, 4) R (1, 2)$

Thus, $(1, 2) R (3, 4)$

$\Rightarrow (3, 4) R (1, 2), \forall (1, 2), (3, 4) \in A \times A$

So, R is symmetric on $A \times A$. **(1)**

Transitive Let $(1, 2), (3, 4), (5, 6) \in A \times A$ such that $(1, 2) R (3, 4)$ and $(3, 4) R (5, 6)$. Then,

$(1, 2) R (3, 4)$

$\Rightarrow \quad 1 + 4 = 2 + 3$

$\quad (3, 4) R (5, 6)$

$\Rightarrow \quad 3 + 6 = 4 + 5$

$\Rightarrow \quad (1 + 4) + 3 + 6 = (2 + 3) + (4 + 5)$

$\Rightarrow \quad 1 + 6 = 2 + 5 \Rightarrow (1, 2) R (5, 6)$

Thus, $(1, 2) R (3, 4)$ and $(3, 4) R (5, 6)$

$\Rightarrow (1, 2) R (5, 6), \forall (1, 2), (3, 4), (5, 6) \in A \times A$

So, R is transitive on $A \times A$. **(1)**

Hence, it is an equivalence relation on $A \times A$.

Equivalence class containing element x of A is given by $\quad [x]_R = \{y : (x, y) \in R\}$

Hence, equivalence class

$[(2, 5)] = \{(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)\}$

24. If the function $f : R \longrightarrow R$ is given by $f(x) = x^2 + 2$ and $g : R \rightarrow R$ is given by $g(x) = \dfrac{x}{x-1}$; $x \neq 1$, then find fog and gof and hence, find $fog(2)$ and $gof(-3)$. **All India 2014**

We have $f(x) = x^2 + 2$ and $g(x) = \frac{x}{x-1}; x \neq 1$

Since, range f = domain g

and range g = domain f

$\therefore$ fog and gof exist.

For any $x \in R$, we have $(fog)(x) = f[g(x)]$

$$= f\left[\frac{x}{x-1}\right] = \left(\frac{x}{x-1}\right)^2 + 2$$

$$= \frac{x^2 + 2\ (x-1)^2}{(x-1)^2} = \frac{x^2 + 2(x^2 + 1 - 2x)}{(x-1)^2}$$

$$= \frac{3x^2 + 2 - 4x}{(x-1)^2}$$

$\therefore fog : R \to R$ is defined by

$$(fog)(x) = \frac{3x^2 - 4x + 2}{(x-1)^2}, \forall x \in R \quad \text{...(i)} \ \mathbf{(1)}$$

For any $x \in R$, we have

$$(gof)(x) = g[f(x)]$$

$$= g[x^2 + 2] = \frac{x^2 + 2}{x^2 + 2 - 1} = \frac{x^2 + 2}{x^2 + 1} \quad \mathbf{(1)}$$

$\therefore gof : R \to R$ is defined by

$$(gof)(x) = \frac{x^2 + 2}{x^2 + 1}, \forall x \in R \quad \text{...(ii)}$$

On putting $x = 2$ in Eq .(i), we get

$$fog(2) = \frac{3 \times (2)^2 - 4(2) + 2}{(2-1)^2} = \frac{3 \times 4 - 8 + 2}{(1)^2}$$

(1)

$$= 12 - 6 = 6$$

On putting $x = -3$ in Eq. (ii), we get

$$(gof)(-3) = \frac{(-3)^2 + 2}{(-3)^2 + 1}$$

$$= \frac{9 + 2}{9 + 1} = \frac{11}{10} \quad \mathbf{(1)}$$

25. If $A = R - \{2\}$ and $B = R - \{1\}$. If $f : A \to B$ is a function defined by $f(x) = \frac{x-1}{x-2}$, then show that f is one-one and onto. Hence, find f^{-1}.

Delhi 2013C

Given, $f(x)=\dfrac{x-1}{x-2}$ and $f: A \to B$, where $A=R-\{2\}$ and $B=R-\{1\}$.

One-one Let $f(x_1)=f(x_2), \forall\, x_1, x_2 \in A$

$$\Rightarrow \quad \frac{x_1-1}{x_1-2}=\frac{x_2-1}{x_2-2} \qquad \textbf{(1/2)}$$

$\Rightarrow \quad (x_1-1)(x_2-2)=(x_2-1)(x_1-2)$

$\Rightarrow x_1x_2-2x_1-x_2+2=x_1x_2-2x_2-x_1+2$

$\Rightarrow \quad -x_1=-x_2 \quad \Rightarrow \quad x_1=x_2$

$\because \quad f(x_1)=f(x_2) \quad \Rightarrow \quad x_1=x_2, \forall\, x_1, x_2 \in A$ **(1)**

Therefore, $f(x)$ is one-one.

Onto Let $y=\dfrac{x-1}{x-2} \Rightarrow xy-2y=x-1$

$\Rightarrow \quad x(y-1)=2y-1$ **(1/2)**

$$\Rightarrow \quad x=\frac{2y-1}{y-1} \qquad \text{...(i)}$$

Since, $x \in R-\{2\}, \forall y \in R-\{1\}$

So, range of $f(x)=R-\{1\}$

$\therefore$ Range = Codomain

Therefore, $f(x)$ is onto. **(1)**

Also, from Eq. (i), we get

$$f^{-1}(y)=\frac{2y-1}{y-1} \qquad [\because x=f^{-1}(y)]$$

$$\Rightarrow \quad f^{-1}(x)=\frac{2x-1}{x-1} \qquad \textbf{(1)}$$

26. Show that the function f in $A=R-\left\{\dfrac{2}{3}\right\}$ defined as $f(x)=\dfrac{4x+3}{6x-4}$ is one-one and onto. Hence, find f^{-1}. **Delhi 2013**

Given, $f(x) = \dfrac{4x+3}{6x-4}$

Let $x_1, x_2 \in A = R - \left\{\dfrac{2}{3}\right\}; x_1 \neq x_2$

One-one Consider, $f(x_1) = f(x_2)$

$$\therefore \quad \frac{4x_1+3}{6x_1-4} = \frac{4x_2+3}{6x_2-4}$$

$$\Rightarrow (4x_1+3)(6x_2-4) = (4x_2+3)(6x_1-4)$$

$$\Rightarrow 24x_1x_2 - 16x_1 + 18x_2 - 12$$
$$= 24x_1x_2 - 16x_2 + 18x_1 - 12$$

$$\Rightarrow \quad -34x_1 = -34x_2 \Rightarrow x_1 = x_2$$

So, f is one-one. **(1½)**

Onto Let $y = \dfrac{4x+3}{6x-4} \Rightarrow 6xy - 4y = 4x + 3$

$$\Rightarrow (6y-4)x = 3 + 4y$$

$$\Rightarrow \quad x = \frac{3+4y}{6y-4} \text{ and } y \neq \frac{4}{6}, \text{ i.e. } y \neq \frac{2}{3}$$

$$\therefore \quad y \in R - \left\{\frac{2}{3}\right\}$$

Thus, f is onto. **(1½)**

Since, f is one-one and onto.

$$\therefore \; x = f^{-1}(y) = \frac{3+4y}{6y-4} \Rightarrow f^{-1}(x) = \frac{3+4x}{6x-4} \quad \textbf{(1)}$$

27. Consider $f : R_+ \to [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible with the inverse f^{-1} of f given by $f^{-1}(y) = \sqrt{y-4}$, where R_+ is the set of all non-negative real numbers.

All India 2013; Foreign 2011; HOTS

> To show that $f(x)$ is an invertible function, we will show that f is both one-one and onto function.

function.

Here, function $f: R_+ \to [4, \infty)$ is given by $f(x) = x^2 + 4$.

Let $x, y \in R_+$, such that $f(x) = f(y)$.

$\Rightarrow \quad x^2 + 4 = y^2 + 4$

$\Rightarrow \quad x^2 = y^2 \Rightarrow x = y$

[$\because$ we take only positive sign as $x, y \in R_+$]

Therefore, f is a one-one function. **(1)**

For $y \in [4, \infty)$,

let $\quad y = x^2 + 4$

$\Rightarrow \quad x^2 = y - 4 \geq 0 \quad [\because y \geq 4]$

$\Rightarrow \quad x = \sqrt{y - 4} \geq 0$

[we take positive sign, as $x \in R_+$]

Therefore, for any $y \in R_+$, there exists $x = \sqrt{y - 4} \in R_+$, such that

$$f(x) = f(\sqrt{y-4}) = (\sqrt{y-4})^2 + 4$$
$$= y - 4 + 4 = y$$

Therefore, f is onto. Thus, f is one-one and onto and therefore, f^{-1} exists. **(1)**

Let us define $g: [4, \infty) \to R_+$, by $g(y) = \sqrt{y-4}$.

Now, $\quad gof(x) = g(f(x)) = g(x^2 + 4)$

$$= \sqrt{(x^2 + 4) - 4} = \sqrt{x^2} = x$$

and $\quad fog(y) = f[g(y)] = f(\sqrt{y-4})$

$$= (\sqrt{y-4})^2 + 4$$
$$= (y - 4) + 4 = y \quad \textbf{(1)}$$

Therefore, $gof = I_{R_+}$ and $fog = I_{[4, \infty)}$

$\Rightarrow \quad f^{-1}(y) = g(y) = \sqrt{y-4}$ **(1)**

NOTE Above fact significantly helps in proving a function f to be invertible by showing that f is one-one and onto, specially when the actual inverse of f is not to be determined.

inverse of f is not to be determined.

28. Show that $f: N \to N$, given by

$$f(x) = \begin{cases} x+1, & \text{if } x \text{ is odd} \\ x-1, & \text{if } x \text{ is even} \end{cases}$$

is bijective (both one-one and onto).

All India 2012

Given function is $f: N \to N$ such that

$$f(x) = \begin{cases} x+1, & \text{if } x \text{ is odd} \\ x-1, & \text{if } x \text{ is even} \end{cases}$$

One-one From the given function, we observe that

Case I When x is odd.

Let $f(x_1) = f(x_2)$

$\Rightarrow x_1 + 1 = x_2 + 1 \Rightarrow x_1 = x_2$

$\because f(x_1) = f(x_2)$

$\Rightarrow x_1 = x_2, \forall\, x_1, x_2 \in N.$

So, $f(x)$ is one-one. **(1)**

Case II When x is even.

Let $f(x_1) = f(x_2)$

$\Rightarrow x_1 - 1 = x_2 - 1 \Rightarrow x_1 = x_2$

$\because f(x_1) = f(x_2)$

$\Rightarrow x_1 = x_2, \forall\, x_1, x_2 \in N.$

So, $f(x)$ is one-one.

Hence, from case I and case II, we observe that, $f(x_1) = f(x_2)$

$\Rightarrow x_1 = x_2, \forall\, x_1, x_2 \in N$

Therefore, $f(x)$ is a one-one. **(1)**

Onto To show $f(x)$ is onto, we show that its range and codomain are same.

From the definition of given function, we observe that

$$f(1) = 1 + 1 = 2$$

$$f(2) = 2 - 1 = 1$$

$$f(2) = 2 - 1 = 1$$
$$f(3) = 3 + 1 = 4$$
$$f(4) = 4 - 1 = 3 \text{ and so on.} \qquad \textbf{(1)}$$

So, we get set of natural numbers as the set of values of $f(x)$.

$\Rightarrow \qquad$ Range of $f(x) = N$

Also, given that codomain $= N$

$$\left[\because f : \underset{\text{domain}}{N} \rightarrow \underset{\text{codomain}}{N}\right]$$

Thus, range = codomain

Therefore, $f(x)$ is an onto function.

Hence, the function $f(x)$ is bijective. **(1)**

29. If $f: R \rightarrow R$ is defined as $f(x) = 10x + 7$. Find the function $g : R \rightarrow R$, such that $gof = fog = I_R$. All India 2011

Given, $f(x) = 10x + 7$

Let $\quad y = 10x + 7 \quad \Rightarrow \quad 10x = y - 7$

$$\Rightarrow \qquad x = \frac{y - 7}{10} \qquad \textbf{(1)}$$

Now, let $\quad g(x) = \dfrac{x-7}{10}$

Then, $gof(x)$ may be written as

$$gof(x) = g[f(x)] = g(10x+7)$$

$$= \frac{10x+7-7}{10} = x \qquad \textbf{(1)}$$

Also, $fog(x)$ may be written as

$$fog(x) = f[g(x)] = f\left(\frac{x-7}{10}\right) = 10\left(\frac{x-7}{10}\right) + 7 \textbf{(1)}$$

$\Rightarrow \qquad fog(x) = x$

Hence, required function $g: R \to R$ is given by

$$g(x) = \frac{x-7}{10} \qquad \textbf{(1)}$$

30. Show that the function $f: W \to W$ defined by

$$f(n) = \begin{cases} n+1, & \text{if } n \text{ is even} \\ n-1, & \text{if } n \text{ is odd} \end{cases}$$

is a bijective function. **All India 2011C**

Do same as Que19.

31. If $f: R \to R$ is the function defined by $f(x) = 4x^3 + 7$, then show that f is a bijection.

Delhi 2011C

The given function is $f : R \to R$ such that

$$f(x) = 4x^3 + 7$$

One-one

Let $f(x_1) = f(x_2), \forall\ x_1, x_2 \in R$

$\Rightarrow$ $4x_1^3 + 7 = 4x_2^3 + 7$

$\Rightarrow$ $4x_1^3 = 4x_2^3 \Rightarrow x_1^3 - x_2^3 = 0$

$\Rightarrow$ $(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = 0$

$[\because a^3 - b^3 = (a-b)(a^2 + ab + b^2)]$

Either $x_1 - x_2 = 0$ ...(i)

or $x_1^2 + x_1x_2 + x_2^2 = 0$ ...(ii)

But Eq. (ii) is not possible as $x_1, x_2 \in R$. **(1/2)**

$\therefore$ $x_1 - x_2 = 0 \Rightarrow x_1 = x_2$

Thus $f(x_1) = f(x_2)$

$\Rightarrow$ $x_1 = x_2,\ \forall\ x_1, x_2 \in R$

Therefore, $f(x)$ is a one-one function. **(1)**

Onto To show that $f(x)$ is an onto function, we show that

Range of $f(x)$ = Codomain of $f(x)$

Given, codomain of $f(x) = R$

Now, let $y = 4x^3 + 7 \Rightarrow 4x^3 = y - 7$

$$\Rightarrow \quad x^3 = \frac{y-7}{4}$$

$$\Rightarrow \quad x = \left(\frac{y-7}{4}\right)^{1/3} \quad \text{...(iii)}$$ **(1/2)**

From Eq. (iii), it is clear that for every $y \in R$, we get $x \in R$.

$\therefore$ Range of $f(x) = R$

Thus, range of $f(x)$ = codomain of $f(x)$

$\Rightarrow$ $f(x)$ is an onto function. **(1)**

Since, $f(x)$ is both one-one and onto, so it is a bijective. **(1)**

32. If Z is the set of all integers and R is the relation on Z defined as $R = \{(a, b) : a, b \in Z$ and $a - b$ is divisible by $5\}$. Prove that R is an equivalence relation.

Delhi 2010; HOTS

The given relation is $R = \{(a, b) : a, b \in Z$ and $a - b$ is divisible by $5\}$. We shall prove that R is reflexive, symmetric and transitive.

(i) **Reflexive** As for any $x \in Z$, we have $x - x = 0$ and 0 is divisible by 5.

$\Rightarrow$ $(x - x)$ is divisible by 5.

$\Rightarrow$ $(x, x) \in R, \forall x \in Z$

Therefore, R is reflexive. **(1)**

(ii) **Symmetric** As $(x, y) \in R$, where $(x, y) \in Z$

$\Rightarrow$ $(x - y)$ is divisible by 5.

[by definition of R]

$\Rightarrow$ $x - y = 5A$ for some $A \in Z$

$\Rightarrow$ $y - x = 5(-A)$

$\Rightarrow$ $(y - x)$ is also divisible by 5

$\Rightarrow$ $(y, x) \in R$

Therefore, R is symmetric. **(1)**

(iii) **Transitive** As $(x, y) \in R$,where $x, y \in Z$

$\Rightarrow$ $(x - y)$ is divisible by 5.

$\Rightarrow$ $x - y = 5A$ for some $A \in Z$

Again, for $(y, z) \in R$, where $y, z \in Z$

$\Rightarrow$ $(y - z)$ is divisible by 5.

$\Rightarrow$ $y - z = 5B$ for some $B \in Z$

Now, $(x - y) + (y - z) = 5A + 5B$

$\Rightarrow \quad x - z = 5(A + B)$

$\Rightarrow$ $(x - z)$ is divisible by 5 for some $A + B \in Z$

Therefore, R is transitive. **(1½)**

Thus, R is reflexive, symmetric and transitive. Hence, it is an equivalence relation. **(1/2)**

NOTE If atleast one of the relation is not satisfied, we do not say it is an equivalence relation.

33. Show that the relation S in the set R of real numbers defined as, $S = \{(a\ b) : a, b \in R$ and $a \leq b^3\}$ is neither reflexive nor symmetric nor transitive. **Delhi 2010; HOTS**

Given relation is

$$S = \{(a, b) : a, b \in R \text{ and } a \leq b^3\}$$

(i) **Reflexive** As $\frac{1}{3} \le \left(\frac{1}{3}\right)^3$, where $\frac{1}{3} \in R$ is not true.

$\Rightarrow \quad \left(\frac{1}{3}, \frac{1}{3}\right) \notin S$

Therefore, S is not reflexive. **(1)**

(ii) **Symmetric** As $-2 \le (3)^3$, where $-2, 3 \in R$ is true but $3 \le (-2)^3$ is not true.

i.e. $(-2, 3) \in S$ but $(3, -2) \notin S$

Therefore, S is not symmetric. **(1)**

(iii) **Transitive** As $3 \le \left(\frac{3}{2}\right)^3$ and $\frac{3}{2} \le \left(\frac{4}{3}\right)^3$, where $3, \frac{3}{2}, \frac{4}{3} \in R$ are true but $3 \le \left(\frac{4}{3}\right)^3$ is not true.

$\Rightarrow \quad \left(3, \frac{3}{2}\right) \in S$ and $\left(\frac{3}{2}, \frac{4}{3}\right) \in S$

but $\quad \left(3, \frac{4}{3}\right) \notin S$ **(1½)**

Therefore, S is not transitive.

Hence, S is none of these, i.e. not reflexive, not symmetric and not transitive. **(1/2)**

NOTE There are certain ordered pairs like (1, 1) for which the relation is reflexive, so it is important to pick example suitably.

34. Show that the relation S in set $A = \{x \in Z : 0 \le x \le 12\}$ given by $S = \{(a, b) : a, b \in Z, |a - b| \text{ is divisible by } 4\}$ is an equivalence relation. Find the set of all elements related to A. **All India 2010**

The given relation is $S = \{(a, b) : |a - b|$ is divisible by 4, where $a, b \in Z\}$

and $\quad A = \{x : x \in Z \text{ and } 0 \le x \le 12\}$

Now, A can be written as

$$A = \{0, 1, 2, 3, \ldots, 12\} \quad \textbf{(1/2)}$$

(i) **Reflexive** As for any $x \in A$, we get $|x - x| = 0$, which is divisible by 4.

$\Rightarrow \quad (x, x) \in S, \forall\, x \in A$

Therefore, S is reflexive. **(1)**

(ii) **Symmetric** As for any $(x, y) \in S$, we get $|x - y|$ is divisible by 4.

[by using definition of given relation]

$\Rightarrow \quad |x - y| = 4\lambda$, for some $\lambda \in Z$

$\Rightarrow \quad |y - x| = 4\lambda$, for some $\lambda \in Z$

$\Rightarrow \quad (y, x) \in S$

Thus, $\quad (x, y) \in S \Rightarrow (y, x) \in S, \forall\, x, y \in Z$

Therefore, S is symmetric. **(1)**

(iii) **Transitive** For any $(x, y) \in S$ and $(y, z) \in S$, we get $|x-y|$ is divisible by 4 and $|y-z|$ is divisible by 4.

[by using definition of given relation]

$\Rightarrow \quad |x-y| = 4\lambda$ and $|y-z| = 4\mu$,

for some $\lambda, \mu \in Z$

Now, $x - z = (x-y) + (y-z)$

$= \pm 4\lambda \pm 4\mu = \pm 4(\lambda + \mu)$

$\Rightarrow \quad |x-z|$ is divisible by 4.

$\Rightarrow \quad (x, z) \in S$

Thus, $(x, y) \in S$ and $(y, z) \in S$

$\Rightarrow \quad (x, z) \in S, \forall\, x, y, z \in Z$

Therefore, S is transitive. **(1)**

Since, S is reflexive, symmetric and transitive, so it is an equivalence relation.

Now, set of all elements related to

$A = \{0,1,2,3,4,5,6,7,8,9,10,11,12\}$ **(1/2)**

35. Show that the relation S defined on set $N \times N$ by $(a, b)\, S\, (c, d) \Rightarrow a + d = b + c$ is an equivalence relation. **All India 2010**

Do same as Que 23.

36. Consider $f : R_+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$, show that f is invertible with $f^{-1}(y) = \left(\frac{\sqrt{y+6}-1}{3}\right)$. **Foreign 2010**

Given function is $f: R_+ \to [-5, \infty)$, such that

$$f(x) = 9x^2 + 6x - 5$$

We shall show that it is both one-one and onto.

One-one

Let $f(x_1) = f(x_2)$, $x, x_2 \in R_+$

$$\Rightarrow 9x_1^2 + 6x_1 - 5 = 9x_2^2 + 6x_2 - 5$$

$$\Rightarrow \quad 9x_1^2 - 9x_2^2 + 6x_1 - 6x_2 = 0$$

$$\Rightarrow \quad 9(x_1^2 - x_2^2) + 6(x_1 - x_2) = 0$$

$$\Rightarrow 9(x_1 - x_2)(x_1 + x_2) + 6(x_1 - x_2) = 0$$

$$\Rightarrow \quad (x_1 - x_2)[9x_1 + 9x_2 + 6] = 0$$

Now, either $x_1 - x_2 = 0$

or $9x_1 + 9x_2 + 6 = 0$

But $9x_1 + 9x_2 + 6 = 0$ is not possible because $x_1, x_2 \in R_+$.

$\therefore \quad x_1 - x_2 = 0 \Rightarrow x_1 = x_2$

Therefore, $f(x)$ is a one-one function. **(1)**

Onto

Let $\quad y = 9x^2 + 6x - 5$

$$\Rightarrow \quad 9x^2 + 6x = y + 5$$

$$\Rightarrow \quad 9\left(x^2 + \frac{6x}{9}\right) = y + 5$$

$$\Rightarrow 9\left(x^2+\frac{2x}{3}+\frac{1}{9}-\frac{1}{9}\right)=y+5$$

$$\Rightarrow \quad 9\left(x+\frac{1}{3}\right)^2-1=y+5$$

$$\Rightarrow 9\left(x+\frac{1}{3}\right)^2=y+6$$

$$\Rightarrow \quad \left(x+\frac{1}{3}\right)^2=\frac{y+6}{9} \Rightarrow x+\frac{1}{3}=\frac{\sqrt{y+6}}{3}$$

[taking positive sign as $x \in R_+$]

$$\Rightarrow \quad x=\frac{\sqrt{y+6}-1}{3} \qquad \textbf{(1)}$$

From above equation, we get that for every $y \in [-5, \infty)$, we have $x \in R_+$.

$\therefore$ Range of $f(x)=[-5, \infty)$

Given, codomain of $f(x)=[-5, \infty)$

Thus, range of $f(x)$ = Codomain of $f(x)$

Therefore, $f(x)$ is an onto function. **(1)**

Since, $f(x)$ is both one-one and onto, so it is an invertible function with

$$f^{-1}(y)=\frac{\sqrt{y+6}-1}{3} \qquad \textbf{(1)}$$

37. If $f: X \to Y$ is a function. Define a relation R on X given by $R=\{(a, b): f(a)=f(b)\}$. Show that R is an equivalence relation on X.

All India 2010C

The given relation is

$$R = \{(a, b) : f(a) = f(b)\}, f : X \to Y$$

(i) **Reflexive** Since, for every $x \in X$, we have

$$f(x) = f(x)$$

[by using definition of R, i.e. $f(a) = f(b)$, $\forall\ a, b \in X$]

$\Rightarrow \quad (x, x) \in R, \forall\ x \in X$

Therefore, R is reflexive. **(1)**

(ii) **Symmetric** Let $(x, y) \in R, \forall\ x, y \in X$

Then, $f(x) = f(y) \Rightarrow f(y) = f(x)$

$\therefore \quad (x, y) \in R, \quad \forall\ x, y \in R$

$\Rightarrow \quad (y, x) \in R, \forall\ x, y \in X$

Therefore, R is symmetric. **(1)**

(iii) **Transitive** Let $x, y, z \in X$

Then $\quad (x, y) \in R$ and $(y, z) \in R$

$\Rightarrow \quad f(x) = f(y), \forall\ x, y \in X \quad \ldots(i)$

and $\quad f(y) = f(z), \forall\ y, z \in X \quad \ldots(ii)$

From Eqs. (i) and (ii), we get

$$f(x) = f(z), \forall\ x, z \in X$$

$\Rightarrow \quad (x, z) \in R, \forall\ x, z \in X$

Thus, $\quad (x, y) \in R$ and $(y, z) \in R$

$\Rightarrow \quad (x, z) \in R, \forall\ x, y, z \in X$

Therefore, R is transitive. **(1½)**

Since, R is reflexive, symmetric and transitive. So, it is an equivalence relation. **(1/2)**

38. Show that a function $f : R \to R$ given by $f(x) = ax + b, a, b \in R, a \neq 0$ is a bijective.

Delhi 2010C

The given function is

$$f(x) = ax + b;\ f: R \to R,\ a, b \in R,\ a \neq 0$$

To show that $f(x)$ is a bijective, we show that $f(x)$ is both one-one and onto.

(i) **One-one** Let $f(x_1) = f(x_2), \forall\ x_1, x_2 \in R$

$\Rightarrow \quad ax_1 + b = ax_2 + b$

$\Rightarrow \quad ax_1 = ax_2 \Rightarrow x_1 = x_2$

Thus, $\quad f(x_1) = f(x_2), \forall\ x_1, x_2 \in R$

$\Rightarrow \quad x_1 = x_2$ **(1½)**

Therefore, $f(x)$ is a one-one function.

(ii) **Onto** Let $\quad y = ax + b$

$\Rightarrow \quad x = \dfrac{y - b}{a}$ ...(i)

From Eq. (i), it is observed that for every $y \in R, x \in R$.

$\therefore \quad$ Range of $f(x) = R$

Also, given codomain $= R$

Thus, range of $f(x) =$ Codomain of $f(x)$

Therefore, $f(x)$ is an onto function. **(1½)**

As $f(x)$ is both one-one and onto, so it is a bijective function. **(1)**

39. Prove that the relation R in set $A = \{1, 2, 3, 4, 5\}$ given by $R = \{(a, b) : |a - b|$ is even$\}$ is an equivalence relation. **Delhi 2009**

The given relation is $R = \{(a, b) : |a - b|$ is even$\}$ defined on set $A = \{1, 2, 3, 4, 5\}$.

(i) **Reflexive** As $|x - x| = 0$ is even, $\forall\ x \in A$.

$\Rightarrow \quad (x, x) \in R, \forall\ x \in A$

Therefore, R is reflexive. **(1)**

(ii) **Symmetric** As $(x, y) \in R \Rightarrow |x - y|$ is even

[by the definition of given relation]

$\Rightarrow \quad |y - x|$ is also even

$[\because |a| = |-a|, \forall\ a \in R]$

$\Rightarrow \quad (y, x) \in R, \forall\ x, y \in A$

Thus, $\quad (x, y) \in R$

$\Rightarrow \quad (y, x) \in R, \forall\ x, y \in A$

Therefore, R is symmetric. **(1)**

(iii) **Transitive** As $(x, y) \in R$ and $(y, z) \in R$

$\Rightarrow \quad |x - y|$ is even and $|y - z|$ is even.

[by using definition of given relation]

Now, $|x-y|$ is even

$\Rightarrow$ x and y both are even or odd

and $|y-x|$ is even

$\Rightarrow$ y and x both are even or odd.

Then two cases arises:

Case I When y is even.

Now, $(x, y) \in R$ and $(y, z) \in R$.

$\Rightarrow$ $|x-y|$ is even and $|y-z|$ is even

$\Rightarrow$ x is even and z is even

$\Rightarrow$ $|x-z|$ is even

[$\because$ difference of two even numbers is also even]

$\Rightarrow$ $(x, z) \in R$ **(1/2)**

Case II When y is odd.

Now, $(x, y) \in R$ and $(y, z) \in R$

$\Rightarrow$ $|x-y|$ is even and $|y-z|$ is even

$\Rightarrow$ x is odd and z is odd

$\Rightarrow$ $|x-z|$ is even

[$\because$ difference of two odd numbers is even]

$\Rightarrow$ $(x, z) \in R$ **(1/2)**

Thus, $(x, y) \in R$ and $(y, z) \in R$

$\Rightarrow$ $(x, z) \in R, \forall\ x, y, z \in A$

Therefore, R is transitive. **(1/2)**

Since, R is reflexive, symmetric and transitive. So, it is an equivalence relation. **(1/2)**

40. If $f : N \to N$ is defined by

$$f(n) = \begin{cases} \dfrac{n+1}{2}, & \text{if } n \text{ is odd} \\ \dfrac{n}{2}, & \text{if } n \text{ is even} \end{cases} \quad \text{for all } n \in N.$$

Find whether the function f is bijective.

All India 2009

The given function is $f : N \to N$ such that

$$f(n) = \begin{cases} \dfrac{n+1}{2}, & \text{if } n \text{ is odd} \\ \dfrac{n}{2}, & \text{if } n \text{ is even} \end{cases}$$

(i) **One-one**

Let

$$f(1) = \frac{1+1}{2} = \frac{2}{2} = 1 \quad \left[\text{put } n = 1 \text{ in } f(n) = \frac{n+1}{2}\right]$$

and $f(2) = \dfrac{2}{2} = 1 \quad \left[\text{put } n = 2 \text{ in } f(n) = \dfrac{n}{2}\right]$

$f(n)$ is not a one-one function because at two distinct values of domain (N), $f(n)$ has same image. **(1½)**

(ii) **Onto** If n is an odd natural number, then $2n - 1$ is also an odd natural number.

Now, $\quad f(2n-1) = \dfrac{2n-1+1}{2} = n \quad \ldots\text{(i)}$

Again, if n is an even natural number, then $2n$ is also an even natural number. Then,

$$f(2n) = \frac{2n}{2} = n \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we observe that for each n (whether even or odd) there exists its pre-image in N.

i.e. Range = Codomain

Therefore, f is onto. **(1½)**

Hence, $f(x)$ is not one-one but it is onto. So, it is not a bijective function. **(1)**

41. Show that relation R in the set of real numbers, defined as $R = \{(a, b) : a \leq b^2\}$ is neither reflexive, nor symmetric nor transitive. **Foreign 2009**

Do same as Que 33.

42. If the function $f : R \to R$ is given by $f(x) = x^2 + 3x + 1$ and $g : R \to R$ is given by $g(x) = 2x - 3$, then find (i) fog and (ii) gof.
All India 2009, 2008C

Given, $f : R \rightarrow R$ such that $f(x) = x^2 + 3x + 1$ and $g : R \rightarrow R$ such that $g(x) = 2x - 3$.

(i) $(fog)(x) = f[g(x)] = f(2x - 3)$

$= (2x - 3)^2 + 3(2x - 3) + 1$

[$\because f(x) = x^2 + 3x + 1$, so replace x by $2x - 3$ in $f(x)$]

$= 4x^2 + 9 - 12x + 6x - 9 + 1$

$= 4x^2 - 6x + 1$ **(2)**

(ii) $(gof)(x) = g[f(x)] = g(x^2 + 3x + 1)$

$= [2(x^2 + 3x + 1)] - 3$

[$\because g(x) = 2x - 3$, so replace x by $x^2 + 3x + 1$ in $g(x)$]

$= 2x^2 + 6x + 2 - 3$

$= 2x^2 + 6x - 1$ **(2)**

43. If the function $f : R \rightarrow R$ is given by $f(x) = \frac{x + 3}{3}$ and $g : R \rightarrow R$ is given by $g(x) = 2x - 3$, then find

(i) fog and (ii) gof. Is $f^{-1} = g$?

Delhi 2009C; HOTS

Given $f: R \to R$ such that $f(x) = \frac{x+3}{3}$ and $g: R \to R$ such that $g(x) = 2x - 3$.

(i) $(fog)(x) = f[g(x)] = f(2x-3) = \frac{(2x-3)+3}{3}$

[$\because$ $f(x) = \frac{x+3}{3}$, so replace x by $2x - 3$ in $f(x)$]

$\Rightarrow \quad (fog)(x) = \frac{2x}{3}$ **(1½)**

(ii) $(gof)(x) = g[f(x)]$

$$= g\left(\frac{x+3}{3}\right) = \left[2\left(\frac{x+3}{3}\right)\right] - 3$$

[$\because g(x) = 2x - 3$, so replace x by $\frac{x+3}{3}$ in $g(x)$]

$$= \frac{2x+6}{3} - 3 = \frac{2x+6-9}{3}$$

$\Rightarrow \quad (gof)(x) = \frac{2x-3}{3}$ **(1½)**

Now, we find f^{-1}. For that, let $y = \frac{x+3}{3}$.

$\Rightarrow \quad 3y = x + 3 \Rightarrow x = 3y - 3$

$\therefore \quad f^{-1}(y) = 3y - 3 \quad [\because x = f^{-1}(y)]$

or $\quad f^{-1}(x) = 3x - 3$

But $g(x) = 2x - 3$.

$\therefore \quad f^{-1} \neq g$ **(1)**

NOTE $f^{-1} = g$ exists, only if $gof = I_R$ and $fog = I_R$.

Binary Operations

1 Mark Questions

1. Let $*: R \times R \to R$ given by $(a,b) \to a + 4b^2$ be a binary operation. Compute $(-5) * (2 * 0)$.
All India 2014C

$$(-5) * (2 * 0) = (-5) * (2 + 4(0)^2) \quad [\because (ab) \to a + 4b^2]$$

$$= (-5) * (2) = -5 + 4(2)^2 = -5 + 16 = 11 \quad \textbf{(1)}$$

2. Let $*$ is a binary operation on the set of all non-zero real numbers, given by $a * b = \frac{ab}{5}$ for all $a, b \in R - \{0\}$. Find the value of x, given that $2 * (x * 5) = 10$.
Delhi 2014

Given, $a * b = \frac{ab}{5}, \forall a, b \in R - \{0\}$ ...(i)

Also given, $2 * (x * 5) = 10$

$$\Rightarrow 2 * \left(\frac{x \cdot 5}{5}\right) = 10 \quad \text{[from Eq. (i)]}$$

$$\Rightarrow 2 * x = 10 \Rightarrow \frac{2x}{5} = 10 \Rightarrow x = 25 \quad \textbf{(1)}$$

3. Let $*$ is a binary operation on N given by $a * b =$ LCM (a, b) for all $a, b \in N$. Find $5 * 7$.
Delhi 2012; Foreign 2008

Given, $a * b = \text{LCM}(a, b), \forall a, b \in N$

$\therefore \quad 5 * 7 = \text{LCM}(5, 7) = 35$

4. Let $*: R \times R \to R$ is defined as $a * b = 2a + b$. Find $(2 * 3) * 4$.
All India 2012

Given, $* : R \times R \to R$

such that $a * b = 2a + b$.

On putting $a = 2$ and $b = 3$, we get

$$(2 * 3) = 2\,(2) + 3 = 4 + 3 = 7$$

$\therefore\ (2 * 3) * 4 = 7 * 4 = 2\,(7) + 4 = 14 + 4 = 18$ **(1)**

5. If the binary operation * on the set of integers Z, is defined by $a * b = a + 3b^2$, then find the value of $8 * 3$. **All India 2012C**

Given, $a * b = a + 3b^2, \forall\ a, b \in z$

On putting $a = 8$ and $b = 3$, we get

$$8 * 3 = 8 + 3 \cdot 3^2 = 8 + 27 = 35$$

6. Let * is a binary operation on set of integers I defined by $a * b = 3a + 4b - 2$, then find the value of $4 * 5$. **All India 2011C**

Given, $a * b = 3a + 4b - 2$

On putting $a = 4$ and $b = 5$, we get

$$4 * 5 = 3(4) + 4(5) - 2$$
$$= 12 + 20 - 2 = 30$$

7. Let * is a binary operation on set of integers I, defined by $a * b = 2a + b - 3$. Find value of $3 * 4$. **Delhi 2011C; All India 2008**

Given, $a * b = 2a + b - 3$

On putting $a = 3$ and $b = 4$, we get

$$3 * 4 = 2(3) + 4 - 3$$
$$= 6 + 4 - 3 = 7$$

8. If the binary operation * on set of integers Z is defined by $a * b = a + 3b^2$, then find the value of $2 * 4$. **Delhi 2009**

Do same as Que 5. **[Ans. 50]**

9. Let * is the binary operation on N given by $a * b = \text{HCF}(a, b)$ where, $a, b \in N$. Write the value of $22 * 4$. **All India 2009**

Given, $a * b =$ HCF of a and b, where a and $b \in N$.

Now, $22 * 4 =$ HCF of 22 and 4

$= \text{HCF of } (2 \times 11) \text{ and } (2 \times 2) = 2$

$\therefore \quad 22 * 4 = 2$ **(1)**

10. If the binary operation *, defined on Q, is defined as $a * b = 2a + b - ab$, for all as $b \in Q$. Find the value of $3 * 4$. **Foreign 2009**

Given, $a * b = 2a + b - ab, \forall\, a, b \in Q$.

On putting $a = 3$ and $b = 4$, we get

$$3 * 4 = 2 \cdot 3 + 4 - 3 \cdot 4$$
$$= 6 + 4 - 12 = -2$$

11. If * is a binary operation on set Q of rational numbers defined as $a * b = \dfrac{ab}{5}$. Write the identity for *, if any. **All India 2009C; HOTS**

Given, binary operation is $a * b = \frac{ab}{5}$.

Let e be the identity element of * on Q.

Then, $a * e = a, \forall a \in Q$

[by definition of identity element]

12. If S is the set of all rational numbers except 1 and * be defined on S by $a * b = a + b - ab$, for all $a, b \in s$.

Prove that

(i) * is a binary operation on S.

(ii) * is commutative as well as associative.

Delhi 2014C

(i) We know that, addition of two rational numbers is a rational number. Also, multiplication of two rational numbers is also a rational number.

Here, a and b are rational numbers other than 1. So, $a+b-ab$ is also a rational number [since difference of two rational numbers is rational number]. So, * is a binary operation on set S. **(1)**

(ii) **Commutative**

$$a * b = a + b - ab = b + a - ba$$

$$\Rightarrow \quad a * b = b * a$$

Hence, * is commutative. **(1)**

Associative $(a * b) * c$

$= (a + b - ab) * c$

$= a + b - ab + c - (a + b - ab)c$

$= a + b + c - ab - bc - ac + abc$ ...(i) **(1)**

and $a * (b * c) = a * (b + c - bc)$

$= a + b + c - bc - a(b + c - bc)$

$= a + b + c - ab - bc - (ac + abc)$ ...(ii)

From Eqs. (i) and (ii), we get

$$(a * b) * c = a * (a * c)$$

Hence, * is associative. **(1)**

13. Consider the binary operations $* : R \times R \to R$ and $o : R \times R \to R$ defined as $a * b = |a - b|$ and $a o b = a$. For all $a, b \in R$. Show that * is commutative but not associative, 'o' is associative but not commutative.

All India 2012

Given $*: R \times R \to R$ such that $a * b = |a - b|$ and $a \circ b = a, \forall\, a, b \in R$.

We have to show that, * is commutative but not associative.

(i) **Commutative**

$$a * b = |a - b|, \forall\, a, b \in R \qquad \text{[given]}$$

and $$b * a = |b - a| \forall\, a, b \in R$$
$$= |-(a - b)|$$
$$= |a - b| \quad [\because |-x| = |x|, \forall\, x \in R]$$

Thus, $a * b = b * a, \forall\, a, b \in R$

Hence, * is commutative. **(1)**

14. Consider the binary operation * on the set $\{1, 2, 3, 4, 5\}$ defined by $a * b = \min\{a, b\}$. Write operation table of operation *.

Delhi 2011

Given, binary operation is $a * b = \min\{a, b\}$ defined on the set $\{1, 2, 3, 4, 5\}$. **(1/2)**

The operation table for operation * is given as follows:

*	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

$[\because 1 * 1 = \min\{1, 1\} = 1, 1 * 2 = \min\{1, 2\} = 1$
$\ldots 5 * 4 = \min\{5, 4\} = 4,$
$5 * 5 = \min\{5, 5\} = 5]$ **(3½)**

15. A binary operation * on the set {0, 1, 2, 3, 4, 5} is defined as $a * b = \begin{cases} a+b, & \text{if } a+b<6 \\ a+b-6, & \text{if } a+b \geq 6 \end{cases}$.

Show that zero is the identity for this operation and each element 'a' of the set is invertible with $6-a$, being the inverse of 'a'.

All India 2011; HOTS

Given $a * b = \begin{cases} a+b, & \text{if } a+b<6 \\ a+b-6, & \text{if } a+b \geq 6 \end{cases}$

The operation table for * is as follows:

*	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

(2)

$[\because 0+0<6 \Rightarrow 0+0=0; 0+1<6 \Rightarrow 0+1=1; \ldots 1+4<6 \Rightarrow 1+4=5; 1+5 \geq 6 \Rightarrow 1+5-6=0; \ldots]$

From table, we note that

(i) $a * 0 = 0 * a = a$. Hence, 0 is the identity for an operation. **(1)**

(ii) $1 * 5 = 0$, $2 * 4 = 0$, $3 * 3 = 0$, $4 * 2 = 0$, $5 * 1 = 0$

Hence, inverse of 1 is 5, i.e.for element a, $6-a$ is its inverse. **(1)**

16. If * is a binary operation on Q, defined by $a*b = 3ab/5$. Show that * is commutative as well as associative. Also, find its identity, if it exists. **Delhi 2010**

Given, binary operation is

$$a * b = \frac{3ab}{5},\ a, b \in Q$$

(i) **Commutative** $a * b = \dfrac{3ab}{5}$ [given]

and $b * a = \dfrac{3ba}{5}$

[by using definition of *]

$\because \quad \dfrac{3ab}{5} = \dfrac{3ba}{5}, \forall\ a, b \in Q$ is true

$\therefore \quad a * b = b * a, \forall\ a, b \in Q$

Therefore, * is commutative. **(1)**

(ii) **Associative**

$$a * (b * c) = a * \left(\frac{3bc}{5}\right) \left[\text{using } a * b = \frac{3ab}{5}\right]$$

$$= \frac{3a\left(\frac{3bc}{5}\right)}{5} = \frac{9abc}{25}$$

and $(a * b) * c = \left(\dfrac{3ab}{5}\right) * c$

$$\left[\because\ a * b = \frac{3ab}{5}, \forall\ a, b \in Q\right]$$

$$= \frac{3\left(\frac{3ab}{5}\right)(c)}{5} = \frac{9abc}{25}$$

Clearly, $a * (b * c) = (a * b) * c,$

$$\forall\ a, b, c \in Q$$

Therefore, * is associative. **(2)**

(iii) **Existence of identity** Let e be the identity element of * on Q. Then, by definition of identity element, we must have

$$a * e = e * a = a, \forall\ a \in Q$$

Let $\quad a * e = a, \forall\ a \in Q$

$\Rightarrow \quad \frac{3ae}{5} = a \quad \left[\because a * b = \frac{3\,ab}{5}\right]$

$\Rightarrow \quad e = \frac{5}{3} \in Q$

$\therefore e = \frac{5}{3}$ is the identity element of * defined on Q. **(1)**

17. If $A = N \times N$ and * is a binary operation on A defined by $(a, b) * (c, d) = (a + c, b + d)$. Show that * is commutative and associative. Also, find identity element for * on A, if any.

Foreign 2010

The given binary operation is

$$(a, b) * (c, d) = (a + c, b + d)$$

defined on $A = N \times N$, we have to show that $*$ is commutative and associative.

(i) **Commutative**

$(a, b) * (c, d) = (a + c, b + d)$,

$\forall (a, b) (c, d) \in N \times N$ [given]...(i)

Also, $(c, d) * (a, b) = (c + a, d + b)$

$\forall (a, b), (c, d) \in N \times N$ ... (ii)

Since, $a + c = c + a, \forall a, c \in N$

and $b + d = d + b, \forall b, d \in N$

From Eqs. (i) and (ii), we get

$(a + c, b + d) = (c + a, b + d)$,

$\forall a, b, c, d \in N$

$\Rightarrow (a, b) * (c, d) = (c, d) * (a, b)$,

$\forall (a, b) (c, d) \in N \times N$

Therefore, $*$ is commutative. **(1)**

(ii) **Associative** $(a, b) * [(c, d) * (e, f)]$

$= (a, b) * (c + e, d + f)$

[using given definition of $*$]

$= (a + c + e, b + d + f)$ ...(i)

Also, $[(a, b) * (c, d)] * (e, f)$

$= (a + c, b + d) * (e, f)$

[using definition of $*$]

$= (a + c + e, b + d + f)$ ...(ii)

From Eqs. (i) and (ii), we get

$(a, b) * [(c, d) * (e, f)] = [(a, b) * (c, d)] * (e, f)$

$\forall (a, b), (c, d), (e, f) \in N \times N$ **(2)**

Therefore * is associative.

(iii) Now, we check the existence of identity of the given operation *.

Let if possible (s, t) be the identity element of the operation *. Then, by definition of identity, we must have

$(a, b) * (s, t) = (a, b), \forall (a, b), (s, t) \in N \times N$

$\Rightarrow \quad (a + s, b + t) = (a, b)$

$[\because (a, b) * (c, d) = (a + c, b + d)$ is given$]$

On equating corresponding elements, we get

$$a + s = a \quad \text{...(i)}$$

and $$b + t = b \quad \text{...(ii)}$$

Eqs. (i) and (ii) are true, when $s = 0$ and $t = 0$

$\therefore \quad (s, t) = (0, 0)$

But $\quad (0, 0) \notin N \times N$

$\Rightarrow$ Identity of the above operation * does not exist as there does not exist any $(s, t) \in N \times N$ such that

$(a, b) * (s, t) = (a, b), \forall (a, b) \in N \times N$ **(1)**

18. If * is the binary operation on N given by $a * b =$ LCM of a and b. Find 20 * 16. Is * (i) commutative and (ii) associative?

All India 2008C

Given $a * b =$ LCM of a and b

LCM of 20 and 16 = 80

$\therefore \quad 20 * 16 = 80$ **(1)**

(i) **Commutative**

$$a * b = \text{LCM of } a \text{ and } b \qquad \text{[given]}$$

and $\quad b * a = \text{LCM of } b \text{ and } a$

$$= \text{LCM of } (b \text{ and } a), \forall\ a, b \in Q$$

and $\quad a * b = b * a, \forall\ a, b \in Q$

Therefore, * is commutative. **(1½)**

(ii) **Associative** $a * (b * c) = a * (\text{LCM of } b \text{ and } c)$

$$[\because\ a * b = \text{LCM of } a \text{ and } b]$$

$\Rightarrow \quad a * (b * c) = \text{LCM of } (a, b \text{ and } c) \quad \ldots\text{(i)}$

and $\quad (a * b) * c = (\text{LCM of } a \text{ and } b) * c$

$\Rightarrow \quad (a * b) * c = \text{LCM of } (a, b \text{ and } c) \quad \ldots\text{(ii)}$

From Eqs. (i) and (ii), we get

$$a * (b * c) = (a * b) * c, \forall\ a, b, c, \in Q$$

Therefore, * is associative. **(1½)**

19. If * is a binary operation on set Q of rational numbers such that $a * b = (2a - b)^2, a, b \in Q$. Find $3 * 5$, $5 * 3$. Is $3 * 5 = 5 * 3$? **Delhi 2008C**

The given binary operation is

$$a * b = (2a - b)^2,\ a, b \in Q$$

$\therefore \quad 3 * 5 = [2(3) - 5]^2$

$$[\text{put } a = 3 \text{ and } b = 5 \text{ in } a * b = (2a - b)^2]$$

$$= (6 - 5)^2 = (1)^2 = 1 \qquad \textbf{(1½)}$$

Also, $\quad 5 * 3 = [2(5) - 3]^2$

$$[\text{put } a = 5 \text{ and } b = 3 \text{ in } a * b = (2a - b)^2]$$

$$= (10 - 3)^2 = (7)^2 = 49 \qquad \textbf{(1½)}$$

Clearly, from above $3 * 5 \neq 5 * 3$ **(1)**